The SU(2) Yang-Mills Fields Must be Massive

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Abstract

The massive Lagrangian for W and Z is deduced from the canonical massless one.

Let W_μ ($\mu \in \{t, x, y, z\}$) be a $SU(2)$ Yang-Mills field:

$$W_\mu = \begin{bmatrix} W_{0,\mu} & W_{1,\mu} - iW_{2,\mu} \\ W_{1,\mu} + iW_{2,\mu} & -W_{0,\mu} \end{bmatrix},$$

for a unitary transformation U:

$$W'_\mu = UW_\mu U^{-1} - \frac{i}{g_2} (\partial_\mu U) U^{-1},$$

$$F_{\mu,\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig_2 [W_\mu, W_\nu],$$

the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} \sum_{\mu,\nu} F^{\mu,\nu} F_{\mu,\nu}.$$

Hence the Euler-Lagrange equations are the following:

$$\sum_\nu \partial^\nu (\partial_\mu W_\nu - \partial_\nu W_\mu - ig_2 [W_\mu, W_\nu]) = 0.$$

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For the components:

$$\begin{array}{l}\sum_\nu \partial^\nu \partial_\nu W_{0,\mu} = 2g_2 \sum_\nu \partial^\nu \left(W_{1,\mu}W_{2,\nu} - W_{2,\mu}W_{1,\nu}\right) + \partial_\mu \sum_\nu \partial^\nu W_{0,\nu},\\ \sum_\nu \partial^\nu \partial_\nu W_{1,\mu} = 2g_2 \sum_\nu \partial^\nu \left(W_{0,\nu}W_{2,\mu} - W_{0,\mu}W_{2,\nu}\right) + \partial_\mu \sum_\nu \partial^\nu W_{1,\nu},\\ \sum_\nu \partial^\nu \partial_\nu W_{2,\mu} = 2g_2 \sum_\nu \partial^\nu \left(W_{0,\mu}W_{1,\nu} - W_{0,\nu}W_{1,\mu}\right) + \partial_\mu \sum_\nu \partial^\nu W_{2,\nu}.\end{array}$$

Let:

$$\begin{array}{l}\alpha_{0,\mu,\nu} = \partial_\nu W_{0,\mu} - 2g_2 \left(W_{1,\mu}W_{2,\nu} - W_{2,\mu}W_{1,\nu}\right),\\ \alpha_{1,\mu,\nu} = \partial_\nu W_{1,\mu} - 2g_2 \left(W_{0,\nu}W_{2,\mu} - W_{0,\mu}W_{2,\nu}\right),\\ \alpha_{2,\mu,\nu} = \partial_\nu W_{2,\mu} - 2g_2 \left(W_{0,\mu}W_{1,\nu} - W_{0,\nu}W_{1,\mu}\right).\end{array} \tag{1}$$

Hence if $\sum_\nu \partial^\nu W_\nu = 0$ then

$$\sum_\nu \partial^\nu \alpha_{0,\mu,\nu} = 0, \sum_\nu \partial^\nu \alpha_{1,\mu,\nu} = 0, \sum_\nu \partial^\nu \alpha_{2,\mu,\nu} = 0.$$

and (see 1 APPENDIX):

$$\begin{array}{c}\sum_\nu \partial_\nu \partial_\nu W_{0,\mu} = -4g_2^2 W_{0,\mu} \sum_\nu W_\nu^2 +\\ +2g_2^2 \sum_\nu \left(W_\mu W_\nu + W_\nu W_\mu\right) W_{0,\nu} + 2g_2 \sum_\nu \left(\alpha_{1,\mu,\nu}W_{2,\nu} - \alpha_{2,\mu,\nu}W_{1,\nu}\right),\end{array} \tag{2}$$

$$\begin{array}{c}\sum_\nu \partial_\nu \partial_\nu W_{1,\mu} = -4g_2^2 W_{1,\mu} \sum_\nu W_\nu^2 +\\ +2g_2^2 \sum_\nu \left(W_\nu W_\mu + W_\mu W_\nu\right) W_{1,\nu} + 2g_2 \sum_\nu \left(W_{0,\nu}\alpha_{2,\mu,\nu} - \alpha_{0,\mu,\nu}W_{2,\nu}\right)\end{array}$$

and

$$\begin{array}{c}\sum_\nu \partial_\nu \partial_\nu W_{2,\mu} = -4g_2^2 W_{2,\mu} \sum_\nu W_\nu^2 +\\ +2g_2^2 \sum_\nu \left(W_\nu W_\mu + W_\mu W_\nu\right) W_{2,\nu} + 2g_2 \sum_\nu \left(\alpha_{0,\mu,\nu}W_{1,\nu} - W_{0,\nu}\alpha_{1,\mu,\nu}\right).\end{array}$$

If

$$\alpha_{\mu,\nu} = \left[\begin{array}{cc} \alpha_{0,\mu,\nu} & \alpha_{1,\mu,\nu} - i\alpha_{2,\mu,\nu} \\ \alpha_{1,\mu,\nu} + i\alpha_{2,\mu,\nu} & -\alpha_{0,\mu,\nu} \end{array}\right]$$

then

$$\begin{array}{c}\sum_\nu \partial_\nu \partial_\nu W_\mu = -4g_2^2 W_\mu \sum_\nu W_\nu^2 +\\ +2g_2^2 \sum_\nu \left(W_\nu W_\mu + W_\mu W_\nu\right) W_\nu - ig_2 \sum_\nu \left(\alpha_{\mu,\nu}W_\nu - W_\nu \alpha_{\mu,\nu}\right)\end{array}$$

and the Lagrangian is:

$$\begin{gathered}\widehat{\mathcal{L}} = \sum_\nu \left(\partial_\nu W_\mu\right)\left(\partial_\nu W_\mu\right) - 4g_2^2 \left(\sum_\nu W_\nu^2\right) W_\mu^2 + \\ + g_2^2 \sum_\nu \left(W_\nu W_\mu + W_\mu W_\nu\right)^2 - \\ - i g_2 \left(\left(\sum_\nu \left[\alpha_{\mu,\nu}, W_\nu\right]\right) W_\mu + W_\mu \left(\sum_\nu \left[\alpha_{\mu,\nu}, W_\nu\right]\right)\right).\end{gathered}$$

It is a lagrangian of a field with mass $2g_2 \left(\sum_\nu W_\nu^2\right)^{\frac{1}{2}}$.

Let B_μ be a Abel gauge field:

$$B'_\mu = B_\mu - \frac{1}{g_1}\partial_\mu \chi$$

Let A_μ and Z_μ are a fields for which:

$$Z_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}}\left(g_2 W_{0,\mu} - g_1 B_\mu\right),\ A_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}}\left(g_2 B_\mu + g_1 W_{0,\mu}\right) \tag{3}$$

and

$$\sum_\nu \partial^\nu \partial_\nu A_\mu = 0. \tag{4}$$

Let us designate:

$$2\left(2\sum_\nu W_\nu^2\right)^{\frac{1}{2}} = \upsilon$$

and

$$2g_2^2 \sum_\nu \left(W_\mu W_\nu + W_\nu W_\mu\right) W_{0,\nu} + 2g_2 \sum_\nu \left(\alpha_{1,\mu,\nu} W_{2,\nu} - \alpha_{2,\mu,\nu} W_{1,\nu}\right) = \Lambda.$$

Hence from (2):

$$\sum_\nu \partial_\nu \partial_\nu W_{0,\mu} = -g_2^2 \frac{\upsilon^2}{2} W_{0,\mu} + \Lambda \tag{5}$$

In this case (see 2 APPENDIX):

$$\sum_\nu \partial^\nu \partial_\nu Z_\mu = -\frac{1}{4}\upsilon^2 \left(g_1^2 + g_2^2\right) Z_\mu -$$

$$-\left(\tfrac{1}{2g_1}\tfrac{1}{g_1^2+g_2^2}\left(g_2^2-g_1^2\right)+\tfrac{1}{\sqrt{g_1^2+g_2^2}}g_1\right)\left(\sum_\nu \partial^\nu\partial_\nu B_\mu+g_1^2\tfrac{\upsilon^2}{2}B_\mu\right)-$$
$$-\left(\tfrac{1}{2g_2}\tfrac{1}{g_1^2+g_2^2}\left(g_2^2-g_1^2\right)-\tfrac{1}{\sqrt{g_1^2+g_2^2}}g_2\right)\Lambda$$

That is Z_μ has got the mass:

$$M_Z=\frac{\upsilon}{2}\sqrt{g_1^2+g_2^2}.$$

1 APPENDIX

From (1):

$$\partial_\nu W_{0,\mu}=\left(2g_2\left(W_{1,\mu}W_{2,\nu}-W_{2,\mu}W_{1,\nu}\right)+\alpha_{0,\mu,\nu}\right), \tag{6}$$

$$\partial_\nu W_{1,\mu}=\left(2g_2\left(W_{0,\nu}W_{2,\mu}-W_{0,\mu}W_{2,\nu}\right)+\alpha_{1,\mu,\nu}\right), \tag{7}$$

$$\partial_\nu W_{2,\mu}=\left(2g_2\left(W_{0,\mu}W_{1,\nu}-W_{0,\nu}W_{1,\mu}\right)+\alpha_{2,\mu,\nu}\right); \tag{8}$$

from (6):

$$\begin{gathered}\partial_\nu\partial_\nu W_{0,\mu}=2g_2\partial_\nu\left(W_{1,\mu}W_{2,\nu}-W_{2,\mu}W_{1,\nu}\right)+\partial_\nu\alpha_{0,\mu,\nu}=\\=2g_2\left(\partial_\nu W_{1,\mu}W_{2,\nu}+W_{1,\mu}\partial_\nu W_{2,\nu}-\partial_\nu W_{2,\mu}W_{1,\nu}-W_{2,\mu}\partial_\nu W_{1,\nu}\right)+\partial_\nu\alpha_{0,\mu,\nu};\end{gathered} \tag{9}$$

hence from (9), (7) and (8):

$$\begin{aligned}\partial_\nu\partial_\nu W_{0,\mu} &= 2g_2\begin{pmatrix}\left(2g_2\left(W_{0,\nu}W_{2,\mu}-W_{0,\mu}W_{2,\nu}\right)+\alpha_{1,\mu,\nu}\right)W_{2,\nu}-\\-\left(2g_2\left(W_{0,\mu}W_{1,\nu}-W_{0,\nu}W_{1,\mu}\right)+\alpha_{2,\mu,\nu}\right)W_{1,\nu}-\\-W_{2,\mu}\partial_\nu W_{1,\nu}+W_{1,\mu}\partial_\nu W_{2,\nu}\end{pmatrix}+\\ &\quad +\partial_\nu\alpha_{0,\mu,\nu};\end{aligned}$$

hence:

$$\begin{gathered}\partial_\nu\partial_\nu W_{0,\mu}=\\=2g_2\begin{pmatrix}2g_2\left(-\left(W_{2,\nu}^2+W_{1,\nu}^2\right)W_{0,\mu}+\left(W_{1,\mu}W_{1,\nu}+W_{2,\mu}W_{2,\nu}\right)W_{0,\nu}\right)+\\+\alpha_{1,\mu,\nu}W_{2,\nu}-\alpha_{2,\mu,\nu}W_{1,\nu}+W_{1,\mu}\partial_\nu W_{2,\nu}-W_{2,\mu}\partial_\nu W_{1,\nu}\end{pmatrix}+\\+\partial_\nu\alpha_{0,\mu,\nu};\end{gathered}$$

and

$$\partial_\nu \partial_\nu W_{0,\mu} = -4g_2^2 \left(W_{2,\nu}^2 + W_{1,\nu}^2 + W_{0,\nu}^2\right) W_{0,\mu} +$$
$$+4g_2^2 \left(W_{0,\mu} W_{0,\nu} + W_{1,\mu} W_{1,\nu} + W_{2,\mu} W_{2,\nu}\right) W_{0,\nu} +$$
$$+2g_2 \left(\alpha_{1,\mu,\nu} W_{2,\nu} - \alpha_{2,\mu,\nu} W_{1,\nu} + W_{1,\mu} \partial_\nu W_{2,\nu} - W_{2,\mu} \partial_\nu W_{1,\nu}\right) + \partial_\nu \alpha_{0,\mu,\nu};$$

if $\sum_\nu \partial_\nu W_\nu = 0$ then:

$$\sum_\nu \partial_\nu \partial_\nu W_{0,\mu} = -4g_2^2 W_{0,\mu} \sum_\nu W_\nu^2 +$$
$$+2g_2^2 \sum_\nu \left(W_\mu W_\nu + W_\nu W_\mu\right) W_{0,\nu} + 2g_2 \sum_\nu \left(\alpha_{1,\mu,\nu} W_{2,\nu} - \alpha_{2,\mu,\nu} W_{1,\nu}\right);$$

2 APPENDIX

From (3):

$$B_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}} \left(g_2 A_\mu - g_1 Z_\mu\right), \; W_\mu^0 = \frac{1}{\sqrt{g_1^2 + g_2^2}} \left(g_1 A_\mu + g_2 Z_\mu\right). \tag{10}$$

and

$$\sum_\nu \partial^\nu \partial_\nu A_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}} \left(g_2 \sum_\nu \partial^\nu \partial_\nu B_\mu + g_1 \sum_\nu \partial^\nu \partial_\nu W_{0,\mu}\right),$$

from (5):

$$\sum_\nu \partial^\nu \partial_\nu A_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}} \left(\begin{array}{c} g_2 \left(\sum_\nu \partial^\nu \partial_\nu B_\mu + g_1^2 \frac{\upsilon^2}{2} B_\mu - g_1^2 \frac{\upsilon^2}{2} B_\mu\right) + \\ +g_1 \left(-g_2^2 \frac{\upsilon^2}{2} W_{0,\mu} + \Lambda\right) \end{array} \right),$$

from (10):

$$\begin{aligned} \sum_\nu \partial^\nu \partial_\nu A_\mu \; &= \; -\frac{\upsilon^2}{2} g_1 g_2 \frac{1}{g_1^2 + g_2^2} \left(2g_1 g_2 A_\mu + \left(g_2^2 - g_1^2\right) Z_\mu\right) + \\ &+ \frac{1}{\sqrt{g_1^2 + g_2^2}} \left(g_1 \Lambda + g_2 \left(\sum_\nu \partial^\nu \partial_\nu B_\mu + g_1^2 \frac{\upsilon^2}{2} B_\mu\right)\right), \end{aligned}$$

from (4):

$$A_\mu = -\left(g_2^2 - g_1^2\right)\frac{1}{2g_1 g_2}Z_\mu + \frac{1}{\upsilon^2\left(g_1 g_2\right)^2}\left(g_1\Lambda + g_2\left(\sum_\nu \partial^\nu\partial_\nu B_\mu + g_1^2\frac{\upsilon^2}{2}B_\mu\right)\right). \tag{11}$$

From (3):

$$\sum_\nu \partial^\nu\partial_\nu Z_\mu = \frac{1}{\sqrt{g_1^2+g_2^2}}\left(g_2\sum_\nu \partial^\nu\partial_\nu W_{0,\mu} - g_1\sum_\nu \partial^\nu\partial_\nu B_\mu\right),$$

from (5):

$$\sum_\nu \partial^\nu\partial_\nu Z_\mu = \frac{1}{\sqrt{g_1^2+g_2^2}}\left(\begin{array}{c} g_2\left(-g_2^2\frac{\upsilon^2}{2}W_{0,\mu} + \Lambda\right) - \\ -g_1\left(\sum_\nu \partial^\nu\partial_\nu B_\mu + g_1^2\frac{\upsilon^2}{2}B_\mu - g_1^2\frac{\upsilon^2}{2}B_\mu\right)\end{array}\right),$$

from (10):

$$\sum_\nu \partial^\nu\partial_\nu Z_\mu = -\frac{\upsilon^2}{2}\frac{1}{g_1^2+g_2^2}\left(g_1^4+g_2^4\right)Z_\mu - g_1 g_2\frac{\upsilon^2}{2}\frac{1}{g_1^2+g_2^2}\left(g_2^2-g_1^2\right)A_\mu +$$

$$+\frac{1}{\sqrt{g_1^2+g_2^2}}\left(g_2\Lambda - g_1\left(\sum_\nu \partial^\nu\partial_\nu B_\mu + g_1^2\frac{\upsilon^2}{2}B_\mu\right)\right)$$

and from (11):

$$\sum_\nu \partial^\nu\partial_\nu Z_\mu = -\frac{1}{2}\frac{\upsilon^2}{2}\left(g_1^2+g_2^2\right)Z_\mu -$$

$$\begin{array}{c} -\frac{1}{2g_1g_2}\frac{1}{g_1^2+g_2^2}\left(g_2^2-g_1^2\right)\left(g_1\Lambda + g_2\left(\sum_\nu \partial^\nu\partial_\nu B_\mu + g_1^2\frac{\upsilon^2}{2}B_\mu\right)\right) + \\ +\frac{1}{\sqrt{g_1^2+g_2^2}}\left(g_2\Lambda - g_1\left(\sum_\nu \partial^\nu\partial_\nu B_\mu + g_1^2\frac{\upsilon^2}{2}B_\mu\right)\right). \end{array}$$